Example 3: Let $V = \ell_2(\mathbb{N})$

$$\ell_2(\mathbb{N}) = \left\{ (\alpha_i)_{i=1}^{\infty} \in \mathcal{S}(\mathbb{R}) \;\middle|\; \sum_{i=1}^{\infty} |\alpha_i|^2 < \infty \right\}$$

We replace V^* with the set of <u>continuous</u> linear functionals from V to $\mathbb{R}$, continuity in the topology determined by the norm $\|\cdot\|_2$.

Lemma: Let V be a vector space over $\mathbb{F}$ and suppose that $f(x) = 0_{\mathbb{F}}$ for some $x \in V$, $\forall f \in V^*$. Then $x = 0_V$.

proof: Let $\mathcal{B} = \{e_s\}_{s \in I}$ be a basis for V. Then $\exists\, n \in \mathbb{N}$, $e_{s_1}, \ldots, e_{s_n} \in \mathcal{B}$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ with

$$x = \sum_{i=1}^{n} \alpha_i e_{s_i}.$$

But we may define as before

$e_s^* \in V^* \ \forall \ s \in I$ by

linearly extending

$$e_s^*(e_t) = \begin{cases} 0, & s \neq t \\ 1, & s = t \end{cases}$$

Then applying $e_{s_j}^*$, $1 \le j \le n$,

$e_{s_j}^*(x) = \alpha_j = 0_{\mathbb{F}}$

if $f(x) = 0_{\mathbb{F}}$ $\forall$ $f \in V^*$.

Hence $x = 0_V$.

$\square$

Double Duals If V is a vector space over $\mathbb{F}$, we define the **double dual** V^{**} of V to be

$$V^{**} = (V^*)^*,$$

the dual of the dual.

Observation: (infinite dimensions)

In finite dimensions, our theorem guarantees us that $V^{**} = V$, isomorphically. In infinite dimensions with restricted duals, this can fail!